Kempner Series

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Alternating Harmonic Series

Harmonic series is the series involving reciprocal of all natural numbers.

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots.$

This series grows without bound, or in mathematical terminology, it diverges (Bandyopadhyay, 2020). A particular modification of the harmonic series is the insertion of negative signs. An *alternating harmonic series* is defined as follows:

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots.$

In the alternating harmonic series, reciprocal of every even number is subtracted, and reciprocal of the odd numbers are added. The alternating harmonic series converges and it converges to $\ln 2$ (approximately, 0.69314718…). The convergence can be readily be obtained from the Taylor series expansion of $\ln(1+x)$. Now, $\ln(1+x)$, for $-1<x\leq 1$, may be expanded as:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots.$$

Substitute *x* as unity in the above expansion and the limit of the alternating harmonic series can be obtained.

The alternating harmonic series is very delicate and should be handled with utmost care. Carefully go through the following derivations.

$$\begin{aligned}
\ln 2 &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}\cdots && \text{(original series)}\\
&= 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}\cdots && \text{(rearrange the even reciprocals)}\\
&= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)\cdots && \text{(group few terms)}\\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}\cdots && \text{(simplification of the groups)}\\
&= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\cdots\right) && \text{(take } \tfrac{1}{2} \text{ common)}\\
&= \frac{1}{2}\ln 2 && \text{(magic ?!?)}
\end{aligned}$$

Therefore, 1 = 2 !

What went wrong? Have we committed any mathematical mistakes? The answer is no. We have not made any silly mathematical errors, but we have made a mathematical blunder. The alternating harmonic series is a *conditionally convergent series,* and we are not permitted to rearrange terms in a conditionally convergent series.

A series that converges even if the negative signs are converted to positive signs is known as an *absolutely convergent series*. For absolutely convergent series, we are allowed to rearrange terms, and the resultant series converges to the original limit. On the other hand, if a convergent series diverges as soon as the negative signs are converted to positive signs, we call it a conditionally convergent series. Though the alternating harmonic series converges, the absolute series (that is the original harmonic series) diverges, and consequently, the alternating harmonic series is a conditionally convergent series. In fact, using a theorem of Georg Friedrich Bernhard Riemann (1826-1866), it is possible to obtain any limit by appropriately arranging terms in the alternating harmonic series (Cowen et al., 1980).

Series without a Particular Digit

Another exciting and curious series can be obtained from the harmonic series if we decide to omit all those terms containing any particular digit. For example, if we skip all those terms containing 9, the resultant series looks as follows:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{18}+\frac{1}{20}+\frac{1}{21}+\cdots+\frac{1}{87}+\frac{1}{88}+\frac{1}{100}+\frac{1}{101}+\cdots$$

If you may think that this restriction excludes only *about one-tenth* of the integers, you are going to make a mistake. It must be astonishing to note that this particular series converges (Kempner, 1914). English-born American mathematician, Aubrey Kempner (1880-1973) studied this series, and this is known as the *Kempner series*.

We follow the proof outlined by Behforooz (1995) to show its convergence. First of all, the terms may be grouped together according to the power of 10. The first group, b_1, consists of the terms from 1 to 1/8; the second group, b_2, consists of the terms from 1/10 to 1/88; and so on. In other words, b_1 consists of the reciprocals of single-digit numbers, b_2 consists of the reciprocals of two-digit numbers, and so on.

$$\left(1+\frac{1}{2}+\cdots+\frac{1}{8}\right)+\left(\frac{1}{10}+\cdots+\frac{1}{88}\right)+\left(\frac{1}{100}+\cdots+\frac{1}{888}\right)+\cdots$$

In general, the nth group, b_n consists of the reciprocals of n-digit numbers. As every term has one as the numerator, we can work only with the denominators without any loss of generality. Now, we need to count the number of terms in each group. This is equivalent to counting the number of n-digit numbers that do not contain 9. The first digit can be chosen in eight ways (any digit may be chosen except 0 and 9). The second digit can be selected in nine ways (any digit may be chosen except 9) and so on. Therefore, an *n*-digit number can be selected precisely $8 \times 9^{n-1}$ ways or b_n contains exactly $8 \times 9^{n-1}$ terms.

Now we can follow the argument, similar to the one used to prove that the harmonic series diverges. As highlighted before, terms of this series are grouped based on the number of digits of the denominator. First eight terms form the first group (starts with 1/1 and ends with 1/8), next 72 terms of the series form the second group (starts with 1/10 and ends with 1/88), following 648 terms constitute the third group (starts with 1/100 and ends with 1/888), and so on. Now, every term of the first group is strictly less than one and therefore, the sum of the numbers in this group is less than 8. Similarly, every term of the second group is less than 1/10, and the sum of the numbers in the entire group is less than $8 \times 9 \times 1/10 = 8 \times (9/10)$. In other words, each term within a group is greater than $1/10^n$, and the sum of all the terms in each group is greater than $8 \times (9/10)^{n-1}$. This forms an *infinite geometric series* with 8 as the initial term and (9/10) as the common ratio. Now, the summation of an infinite geometric converges to a finite number, whenever the common ratio is less than unity. Using the property that if a series with higher values converges, the series lower values also converges, we can establish that Kempner series converges too. This argument can be mathematically written as follows:

$$\left(\overbrace{1+\cdots+\frac{1}{8}}^{8\,terms}\right)+\left(\overbrace{\frac{1}{10}+\cdots+\frac{1}{88}}^{8\times 9\,terms}\right)+\left(\overbrace{\frac{1}{100}+\cdots+\frac{1}{888}}^{8\times 9^2\,terms}\right)+\cdots+\left(\overbrace{\frac{1}{10^n}+\cdots+\frac{1}{\underbrace{88...8}_{n-times}}}^{8\times 9^{n-1}\,terms}\right)+\ldots$$

$$<\left(\overbrace{1+\cdots+1}^{8\,terms}\right)+\left(\overbrace{\frac{1}{10}+\cdots+\frac{1}{10}}^{8\times 9\,terms}\right)+\left(\overbrace{\frac{1}{100}+\cdots+\frac{1}{100}}^{8\times 9^2\,terms}\right)+\cdots+\left(\overbrace{\frac{1}{10^n}+\cdots+\frac{1}{10^n}}^{8\times 9^{n-1}\,terms}\right)+\ldots$$

$$=8+\frac{8\times 9}{10}+\frac{8\times 9^2}{10^2}+\frac{8\times 9^3}{10^3}+\ldots+\frac{8\times 9^n}{10^n}+\ldots=8\times\frac{1}{1-(9/10)}=80$$

This series actually converges extremely slowly to 22.92067661926415034816... (Baillie, 1979).

The same proof can be extended to show that the harmonic series, reduced by omitting all those terms containing an 8, or a 7, ..., or 1, converges. In the case with the digit 0, a slight adjustment in the proof is required to obtain the result (Hardy and Wright, 1979).

It may be interesting to note a critical consequence of the previous result. The number of undeleted terms remaining in b_n (i.e., *n*-digit numbers) is seen to be precisely $8 \times 9^{n-1}$, and there are a total of $9 \times 10^{n-1}$ *n*-digit numbers. Therefore,

$$\frac{Number\ of\ n\text{-}digit\ numers\ without\ 9}{Total\ number\ of\ n\text{-}digit\ numers}=\frac{8\times 9^{n-1}}{9\times 10^{n-1}}=\left(\frac{8}{9}\right)\left(\frac{9}{10}\right)^{n-1}.$$

For 10-digit numbers, about 34% of them do not contain 9. If we think of 100-digit numbers, less than 0.003% of them do not include 9. Note that as *n* grows beyond bound, the above ratio tends to zero. Consequently, over a broad range of natural numbers, *almost all* the numbers do contain the digit 9. Over the totality of natural numbers, the probability that a randomly chosen number does not contain a 9 is negligible. However, the same can be stated for every other digit 0, 1, 2, ..., 8. Thus, we can conclude with a probabilistic certainty that a randomly chosen natural number contains every digit at least once. This is not so surprising considering that *most* natural numbers do contain millions of digits.

You may now be able to appreciate the fact that, instead of a single digit, if we decide to delete all those terms containing a particular number (say '27' or '123456' or any other number that you wish), the series still converges (Schmelzer and Baillie, 2008).

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