Magic Hexagon

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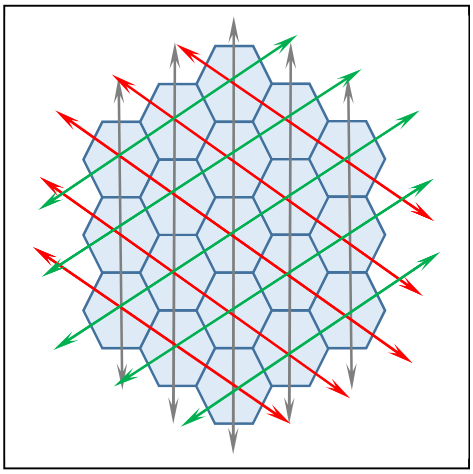
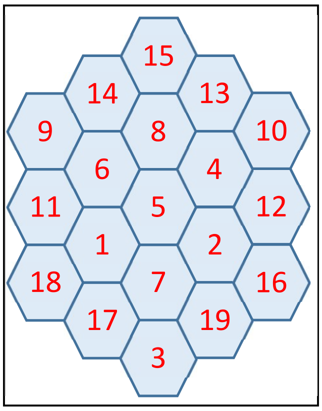
In 1910, Clifford W. Adams got interested in a puzzle published in a local newspaper (Trigg, 1964). The problem was to place the first nineteen natural numbers in the vacant cells of a hexagon such that the sums of the different diagonals and verticals (total fifteen in number, as shown in the box →) are equal.



A hexagon is a six-sided polygon. Nineteen hexagonal cells are placed together to form a larger hexagonal structure (see the box →). There is a central hexagonal cell, the first order hexagon. This central cell is surrounded by six hexagonal cells to form the second-order hexagon. An additional twelve hexagonal cells are placed around to form the third-order hexagon. The puzzle asks to put the first 19 natural numbers in these 19 hexagonal cells such that the sum of the numbers in different directions are identical. The sum of the first nineteen natural number is 190. Five parallel rows (red, green, or grey lines, as shown in the box above ↑) encompasses all these natural numbers. Therefore, the *magic sum*, i.e., the identical sum expected in all directions for the *magic hexagon*, should be 190/5 = 38.

Adam used to work as a freight handler and clerk with the Reading Railroad in Philadelphia and made several attempts to find a solution to the puzzle (Trigg, 1964). Finally, after an intense search for 47 years, he discovered a solution and wrote it in a piece of paper. Unfortunately, he misplaced the paper, and finally, in December 1962, he rediscovered it by relocating the lost solution (Gardner, 1971).



Adams sent a copy of his magic hexagon to Martin Gardner. Gardner initially thought that this must be similar to some *magic square* and must also be well known. However, he could not locate such a magic hexagon in any popular work on recreational mathematics. He forwarded a copy of this hexagon to Charles W. Trigg, a mathematician from San Diego, California, if he knew of any reference to magic hexagons (Gardner, 1971).

Trigg first proved that hexagon of any other size could not be magic (other than the trivial solution of single hexagon containing only 1). Then he demonstrated that this is unique (without counting reflection or rotation). On 25th April of 1963, he communicated these results to Gardner, and the entire story was published in Scientific American in its August issue of 1963 (Trigg, 1964). Later on, the uniqueness of this magic hexagon was confirmed independently by William Daly, G. W. Anderson, and Eduardo Esperón (Trigg, 1964).

We have already mentioned that the size of this magic square is unique. Let's prove this result. It may be easily seen that the first-order magic hexagon is unique and trivial. It contains only one hexagonal cell with 1 inside.

Let us construct a second-order magic hexagon. It contains a total of 7 cells. Now, mark three cells on edge as a, b, and c (with b on the corner cell). It can be easily concluded that this hexagon can be magic, only when $a + b = b + c$, or when $a = c$. Therefore, all seven cells cannot contain 7 different natural numbers, and hence, a second-order magic hexagon cannot exist.

Let us now consider a general order-n magic hexagon. For $n = 1, 2, 3, \ldots$, an order-n magic hexagon contains 1, 7, 19, 37, 61, 91, 127, 169, ... (A003215 in OEIS) number of cells. These numbers are called *hex numbers* and mathematically can be represented as $3n^2 - 3n + 1$. Therefore, a general order-n hexagonal arrangement contains natural numbers 1 to $3n^2 - 3n + 1$ to occupy various cells. The addition of m consecutive natural numbers, starting from 1, is expressed by the formula: $m(m+1)/2$. Numbers of the form $m(m+1)/2$, are also called *triangular numbers*, as you can arrange them to form a triangle. Therefore, the total of all the numbers in a general order-n hexagonal arrangement can be calculated as:

$$s = \frac{1}{2}(3n^2 - 3n + 1)(3n^2 - 3n + 2) = \frac{9n^4 - 18n^3 + 18n^2 - 9n + 2}{2}$$

Note that a general order-n hexagonal arrangement has $2n - 1$ rows in each direction. To have the magical property, the sum of each row must be the same. In other words, the total of all these numbers must be divisible by $2n - 1$ and the magic constant, M, may be calculated as:

$$M = \frac{s}{2n-1} = \frac{9n^4 - 18n^3 + 18n^2 - 9n + 2}{2(2n-1)}$$

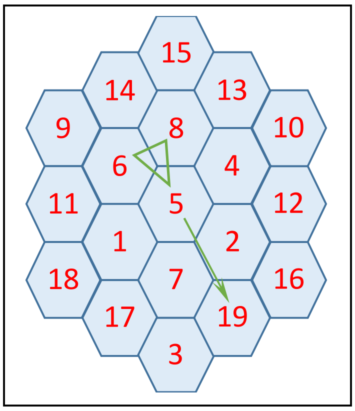
The above equation can be further simplified as:

$$32M = 72n^3 - 108n^2 + 90n - 27 + \frac{5}{(2n-1)}$$

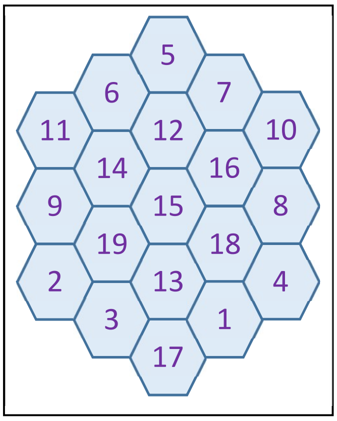
Now, the magic constant, M, can be an integer only if, $2n - 1$ divides 5. Since 5 is a prime number, this possible only if $2n - 1$ is either 1 or 5. This implies that n can either be 1 or 3.

Now $n = 1$ corresponds to the first-order magic hexagon, a unique but non-interesting one. On the other hand, $n = 3$ corresponds to the third-order magic hexagon. As n cannot be of any other value, a magic hexagon of any other size cannot exist.

Roderick Collar had shown a delightful property of the magic hexagon (Tapson, 1987). Choose any two adjacent cells from the second-order layer. They form a triangle with the central cell. Now, the total of all these numbers is represented in an outer cell that the triangle points out. For example, 6, 8, and 5 form a triangle, and this triangle points towards an outer cell that contains the sum of these numbers, i.e., 19 (= 6 + 8 + 5), as demonstrated in the adjacent box (→). You may explore other possible triangles. A simple mathematical proof of this property can be found in the book by Honsberger (1973).

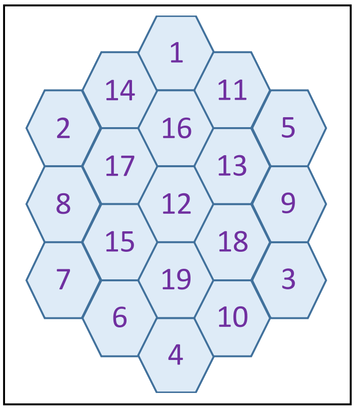


Let us look at the history of the magic hexagon. This unique magic hexagon was rediscovered independently by various researchers. In 1888, von Haselberg, of Stralsund proposed a problem related to the unique magic hexagon in the German magazine "*Zeitschrift fur mathematischen und naturwissenschaftlichen Unterricht*" (Gardner, 1988). Till now, this is the earliest known reference to the magic hexagon. In later years, it was discovered independently by William Redcliffe in 1895 (Tapson, 1987), by Martin Kühl around 1940 (Gardner, 1971), and by Vickers in 1958 (Vickers, 1958).



We can create another interesting representation, by replacing every number x in the original magic hexagon by $20 - x$. Now, this hexagonal arrangement is no longer magic (see the box ←). However, it has a fascinating property. Adding the numbers in any direction, containing only three cells, equals to 22. Similarly, the total of the numbers in any direction, containing four and five cells, equals to 42 and 62. For example, 11 + 6 + 5 = 22; 6 + 12 + 16 + 8 = 42; 5 + 12 + 15 + 13 + 17 = 62. Check it yourself for other directions. Six directions are containing either three or for cells, and only three directions contain five cells. This particular hexagon may be called *22-42-62 hexagon*. Note that this is unique as the original magic hexagon is unique.

In general, if the first 19 natural numbers are arranged in the 19 hexagonal cells in such a way that the addition of all the numbers in any direction, containing three, four, and five cells, are A, B, and C, we can call it an *A-B-C hexagon*. You can easily verify that $2A + 2B + C = 190$. There are many hexagons possible for different values of A, B, and C. When $A = B = C$, it produces the original magic hexagon. If only two of them are equal, we may call it semi-magic. The lowest value of A for which a solution exists is 17. Without



accounting for rotation and reflection, there are two different solutions for a 17-50-56 hexagon. On the other hand, solutions for 17-52-52 and 17-53-50 hexagons are unique. No other solution exists for A = 17. The unique 17-52-52 semi-magic hexagon is shown in the adjacent box (←). Interested readers are encouraged to check what happens when every number x in this semi-magic hexagon is replaced by $20 - x$.

References

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