Harmonic Series

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One of the fascinating infinite series is the harmonic series, a series involving reciprocal of all natural numbers:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots.$$

Some of the remarkable and curious properties of this series are discussed here.

The harmonic series diverges. In other words, its value increases without bound. This can be demonstrated as follows. Several terms of the series may be grouped together; for example, first nine terms form the first group (stars with 1/1 and ends with 1/9), next 90 terms of the series form the second group (stars with 1/10 and ends with 1/99), following 900 terms constitute the third group (stars with 1/100 and ends with 1/999), and so on. In mathematical notation, each group, consisting of 9×10^i terms, starts with $1/10^i$ and ends with $1/(10^{i+1}-1)$. Now, every term of the first group is strictly greater than 1/10, and therefore, the sum of all the terms in the group is greater than 9 × (1/10) = 9/10. Similarly, every term of the second group is greater than 1/100, and the sum of all the numbers in the entire group is greater than 90 × (1/100) = 9/10. In other words, each term within a group is greater than $1/10^{i+1}$, and the sum of all the terms in each group is greater than 9/10. Since summation of infinitely many positive numbers diverges, the harmonic series itself grows beyond a limit. Here, we are using the property that if a lesser series diverges, the higher series diverges too. This argument can be symbolically written as follows:

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i} &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots \\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{9}\right)+\left(\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{99}\right)+\left(\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{999}\right)+\cdots \\ &> \left(\frac{1}{10}+\frac{1}{10}+\cdots+\frac{1}{10}\right)+\left(\frac{1}{100}+\frac{1}{100}+\cdots+\frac{1}{100}\right)+\left(\frac{1}{1000}+\cdots+\frac{1}{1000}\right)+\cdots \\ &= \frac{9}{10}+\frac{90}{100}+\frac{900}{1000}+\cdots=\frac{9}{10}+\frac{9}{10}+\frac{9}{10}+\cdots\to\infty\end{aligned}$$

This particular proof follows the argument outlined by Honsberger (1976). Readers are encouraged to read the paper by Kifowit and Stamps (2006) for additional proofs using different concepts and approaches.

The harmonic series, though divergent, grows extremely slowly. For example, the sum of a quarter of a billion terms is still less than 20. It takes exactly 272400600 terms to pass

20. In fact, the sum of the first 272400599 terms is approximately 19.9999999979, and adding 1/272400600, the total is approximately 20.0000000016 (Boas and Wrench, 1971). To surpass 100, it is necessary to add up more than 15 million trillion trillion trillion terms (i.e., more than 15×10^{42} terms). To exceed natural numbers 1, 2, 3, 4, ..., the harmonic series need at least following number of terms: 1, 4, 11, 31, 83, 227, 616, 1674, 4550, 12367, 33617, 91380, 248397, ... (A004080 in OEIS).

As the harmonic series diverges, it goes beyond every natural number. Except for the first term, surprisingly, it manages to avoid every integer in doing so (Osler, 2012). This surprising fact can be proved without much difficulty. Let us take the partial sum of the series up to first n terms ($n > 1$) and denote it as $H_n = \sum_1^n \left(\frac{1}{i}\right)$. As n is more than 2 ($n > 2$), there are plenty of even denominators in this partial series. Choose the number k ($\leq n$) such that it is of the form 2^m with m being the highest (for example, if $n = 10$, choose k as 8; if $n = 20$, choose k as 16, and so on). It may not be difficult to appreciate that for any given n, there is a unique number of this form. Let us now factor all the denominators into a product of prime factors and calculate the least common factor for all these denominators. Multiply both the numerator and the denominator for every fraction by a number to make the denominator equals the least common factor for all these denominators. Recall your school days; the identical procedure was taught for summing fractions. All fractions other than $1/k$ are multiplied by two or some higher power of it, and eventually, they become even. On the other hand, as k is the highest power of 2, its numerator is multiplied by only the odd prime factors, and it becomes an odd number. When we are going to sum up all the numerators to calculate H_n, it has to be an odd number (recall that the sum of an even number with an odd number is always odd). Now the fraction, with an odd numerator and an even denominator, cannot be an integer. The partial sum of the harmonic series, except for the first term, can never be an integer.

Note that $H_1 = 1$, $H_2 = 1.5$, and $H_6 = 2.45$. Interestingly, other than these three partial sums, all other partial sums are always infinitely recurring. See the article on cyclic numbers (Bandyopadhyay, 2020) for finite recurring decimals. This fact can be proved based on the observation of Joseph Bertrand (1822-1900). In 1845, the French mathematician Joseph Bertrand conjectured that for every positive integer n (> 1), there exists at least one prime number between n and $2n$. Five years later, the renowned Russian mathematician Pafnuty Cheybychev (1821-1894) proved this (Crilly, 1989).

What happens when we remove all the odd denominators from the harmonic series? The resultant series can be manipulated as follows:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\right)$$

and the resultant series diverges. You may wonder what happens to the other half. We may express the other part as

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\cdots>1+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots=\frac{1}{2}+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\right)$$

Without any doubt, this series also increases without bound. In the next section, we are further truncate the original harmonic series by removing all the terms with composite denominators.

Series with Primes only

We all know that there exist infinitely many prime numbers. Euclid proved this fact, in his *Elements*, in an elegant way. We prove this by showing a contradiction to the hypothesis that there are only finitely many prime numbers. Let the finitely many prime numbers are denoted as $p_1 = 2 < p_2 = 3 < ... < p_k$. We can define a number P as one more than the product of all these primes, $P = p_1 p_2 ... p_k + 1$. Let p be a prime divides P; then p cannot be any of $p_1, p_2, ..., p_k$, otherwise p would divide the difference $P - p_1 p_2 ... p_k = 1$, which is impossible. Therefore, p is still another prime, and p_1, p_2, ..., p_k would not be all the primes, contradicting the initial hypothesis of finitely many prime numbers. In other words, this proves the existence of infinitely many primes.

If all the terms with composite denominators are deleted from the harmonic series, a drastically reduced series is obtained with terms only with prime number denominators.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots$$

It is surprising to note that this infinite series also diverges (Hardy and Wright, 1979). As expected, the sum of the reciprocals of the primes diverges very very slowly. For example, the sum of the reciprocals of the first one million primes is only 2.887289… Similar to the harmonic series, the partial sums of the reciprocals of the primes are never an integer. Interestingly, the proof is even simpler. If for any set of prime numbers p_1, p_2, p_3, …, p_k, the sum of the reciprocals of these primes is an integer, then

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\cdots+\frac{1}{p_k}=n$$

This may be expressed as

$$\frac{1}{p_1}=n-\frac{1}{p_2}-\frac{1}{p_3}\cdots-\frac{1}{p_k}=\frac{a}{p_2p_3p_4\cdots p_k}$$

for some integer a and by rearranging $ap_1 = p_2p_3p_4...p_k$. This leads to a contradiction because p_1 cannot divide the product of other primes. Therefore, by contradiction, we proved that the partial sums of the reciprocals of the primes are never an integer.

If p and $p + 2$ are both primes, they are called *twin primes*. For example, 3 and 5; 11 and 13, 1019 and 1021, etc. It may be curious to note that 5 is the only prime that appears in two sets: (3, 5) and (5, 7). If we modify the series with prime numbers with only twin primes and account for 5 twice, the resultant series is as follows:

$$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)\cdots$$

In 1919, Norwegian mathematician Viggo Brun (1885-1978) established the convergence of the series (Shanks and Wrench, 1974). The series converges to the number 1.9021605824…, known as Brun's constant. Thomas Nicely calculated the numerical value of the Burn's constant, and in the process, he unearthed the well-known and infamous flowing point error associated with Intel Pentium Processors (Cipra, 1995).

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