Horror Infinity

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23 February 2020

The Concept of Infinity

Is there a *number* that is larger than any *real number*? Philosophers and mathematicians have always been fascinated by the concept of *infinity*. Infinity is not a number; instead, it is a process that goes on, and on, and on. Our experiences in the finite world may not be reliable to understand infinity. In the language of Fischbein (1987), "when dealing with actual infinity—namely with infinite sets—we are facing situations which may appear intuitively unacceptable … we are intuitively not equipped to deal with actually given infinite sets. Their logic is not our logic, which is rooted in our practical experience." Infinity may only be understood and appreciated through a *limiting* process.

To understand the idea of limit, one of the most fundamental concepts of mathematical analysis, let's consider the sequence of numbers a_1, a_2, a_3, …, a_n, …. By saying that this sequence *tends to a limit 'a' as 'n' tends to infinity*, it is meant that as n grows larger and larger without bound, the terms of the sequence get nearer and nearer to a. It suggests that for any predetermined small positive number, there exists a positive number N such that whenever n is larger than N, the absolute difference between a_n and the limiting value a becomes less than the predetermined number. Symbolically, we represent it as $\lim_{n\to\infty} a_n = a$.

Let us consider the sequence 0, 3/2, 2/3, 5/4, 4/5, …, whose general term is $a_n = 1 + (-1)^n/n$. As n grows larger and larger, the terms get closer and closer to 1. Suppose we choose a small number, say 0.001, then N is 1000, and whenever n is greater than 1000, $|1 - a_n|$ is always be less than 0.001. We write, $1 + (-1)^n/n \to 1$ as $n\to\infty$ and in abbreviated notation:

$$\lim_{n\to\infty}\left[1+\frac{(-1)^n}{n}\right]=1.$$

In modern mathematical language, for a given ε, if there exists a δ such that whenever $0 \le |z - x| \le \delta$, we have $|f(z) - l| \le \varepsilon$, we abbreviated it as $\lim_{z\to x} f(z) = l$.

If a sequence of numbers grows without bound, as $n\to\infty$, we express this by writing $\lim_{n\to\infty} a_n = \infty$, although strictly speaking, the sequence does not have a limit. A limit—if it exists—must be a definite real number, and infinite is not a finite real number. We say that the sequence diverges.

There may be sequences, which are oscillatory in nature. For example, the sequence $a_n = (-1)^n$. As n grows, the sequence oscillates between +1 and –1. Similarly, the series 1 – 1 + 1 – 1 + 1 – ... also oscillates between 1 and 0. With the concept of limit discussed earlier, we say that the sequence and the series do not converge, and the limits for them do not exist. Such a series is called divergent.

The divergent series 1 – 1 + 1 – 1 + 1 – ... obtained by setting $x = 1$ into the *identity* $1/(1 + x) = 1 - x + x^2 - x^3 + x^4 - \ldots$ leads to a sum of 1/2. In the eighteenth century, this paradoxical result provoked metaphysical and theological discussion to a significant extent. Strictly speaking, a substitution like this is not permitted, as the series is convergent only for $-1 \leq x < 1$, and the series diverges for all other values of x. This kind of *algebraic analysis*—so brilliantly applied by Euler and used by most of the eighteenth-century mathematicians—was accepted as an article of faith that what is accurate for a convergent series is also accurate for a divergent series. The notion of limit was only delineated in the nineteenth century. Augustin Louis Cauchy (1789-1857) and Karl Wilhelm Theodor Weierstrass (1815-1897) were instrumental in developing the modern concept of limit.

It is interesting to note that we cannot represent the symbol of infinity ∞, as an ordinary number. Mathematically speaking, expressions such as $0/0$, ∞/∞, $0.\infty$, $\infty\text{-}\infty$, 0^0, ∞^0, 1^∞, etc. are not defined and are called *indeterminate forms*. These expressions have no preassigned value; they can only be evaluated through a limiting process. The final result depends on the particular limiting process involved in its evaluation.

Probably, Archimedes of Syracuse (ca. 287-212 BC) was the first mathematician to use the concept of infinity fruitfully. He solved the problem of finding the area of a parabola by dividing the sector into a series of triangles whose areas decrease in a geometric progression. By continuing this progression on and on, he could make the triangles fit the parabola as closely as he pleased—*exhaust* it—as he expressed so. In modern terms, the combined area of the triangles approaches a limit as the number of triangles tends to infinity. Archimedes was cautious about formulating his solution in terms of finite sums only without mentioning the word infinity in his argument. This was primarily because the Greeks were horrified with the concept of infinity, what they termed *horror infiniti*.

Zeno's Paradoxes

Around the fourth century BC, philosopher Zeno of Elea came up with four paradoxes to demonstrate the inability of mathematics to cope up with the concept of infinity. These paradoxes are *the dichotomy paradox*, *the Achilles and the tortoise paradox*, *the arrow paradox*, and *the stadium paradox*.

In the first paradox, known as *the dichotomy paradox*, Zeno argued to show that the motion is impossible (Papa-Grimaldi, 1996). For a runner to cover a given distance *d*, the runner must first cover half the required distance, then half of the remaining distance, then half of that, and so on, *ad infinitum*. As this process involves an infinite number of steps, Zeno argued, the distance couldn't be travelled in totality. In modern notation, the runner covers a total distance given by the infinite geometric series:

$$\frac{d}{2}+\frac{d}{4}+\frac{d}{8}+\frac{d}{16}+\cdots+\frac{d}{2^n}+\cdots.$$

The series keeps growing and approaches *d*. The series converges to limit *d* as the number of terms grows larger, beyond bound. Thus, the total distance covered is precisely *d*. Now the time intervals it takes the runner to travel these partial distances—assuming a constant speed for the runner—also follow similar infinite, but convergent series, and hence, the entire length can be covered in finite time. This resolves the paradox.

The second paradox, known as *the Achilles and the tortoise paradox*, proceeds in a similar line (Papa-Grimaldi, 1996). Achilles is to catch a tortoise that has a head start. At any point in time, suppose Achilles is at a position '*A,'* and tortoise is at another position '*B*' (*B* is ahead of *A*). Achilles needs some time to reach *B* from *A*, and during this time, tortoise moves forward to another position '*C*.' Zeno argued that Achilles could never be able to catch up with the tortoise. The resolution is similar to the dichotomy paradox through a convergent series.

The other two lesser-known paradoxes are *the arrow paradox* and *the stadium paradox*. The arrow paradox is related to the perceiving motion with the argument that at any point in time, a moving arrow must be at rest (Papa-Grimaldi, 1996). The stadium paradox arises from the assumption that space and the time can be divided only by a definite amount. Zeno paradoxically argued that double the time is equal to half the time (Papa-Grimaldi, 1996).

The Greeks, however, did not subscribe to such reasoning that a sum of infinitely many numbers might have a finite value. To achieve the desired accuracy, they can add up as many terms as necessary, but the idea of extending that procedures to infinity intrigued them great intellectual distress. Their fear of the infinity led them to bar it from their mathematical system.

References

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