Cyclic Number

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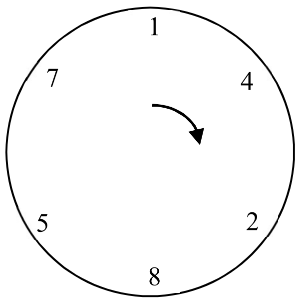
A *cyclic number* has an unusually interesting property. If you multiply a cyclic number, by 1 through n (where n is the number of digits of the cyclic number), these products contain the same n digits of the initial number in exactly the identical cyclic order. For example, 142857 is a cyclic number. Digits of 142857 always appear in the same order, but cycle around, when multiplied by 1 through 6.

$142857 \times 1 = 142857$
$142857 \times 2 = 285714$
$142857 \times 3 = 428571$
$142857 \times 4 = 571428$
$142857 \times 5 = 714285$
$142857 \times 6 = 857142$



The reciprocal of 7 (that is 1 divided by 7) is a repeating decimal with a period length of 6. That is 1/7 = 0.142857 142857... and we express the repeating decimals by putting a bar on top of it, i.e., $1/7 = 0.\overline{142857}$. Similarly, $1/142857 = 0.\overline{000007}$. In this sense, we may say that 7 generates cyclic number 142857.

Note that reciprocal of 7 has repeating decimals with a period length of 6. In fact, no fraction with d as its denominator can have a repeating period longer than $d - 1$. Let's understand why. If you divide one by d, it can have a maximum of $d - 1$ possible remainders during each step of the division process. When a remainder is repeated, the period also starts repeating. Therefore, every fraction with a denominator d must have a period length of $d - 1$ or smaller. Note that the above result does not imply that the period length cannot be smaller than $d - 1$; it only suggests that it cannot be more than $d - 1$. It has been proved that the maximum-length period, the repeating period of $d - 1$ for the reciprocal of d, is achieved only when d is prime. Therefore, cyclic numbers are generated by primes with maximum period length. That is, if $1/p$, with a prime p, generates a repeating decimal with a period of $p - 1$, then the decimal expansion of $1/p$ contains a cyclic number. Such primes are called cyclic primes or full-period primes.

It is not difficult to see why such maximum length periods are cyclic. Let's consider 3/7 as an example. As all possible remainder appears in dividing one by 7, dividing 3 by 7 is equivalent to starting the entire cyclic process, only at a different place. This is certain to get the same cyclic order of digits on the periods of the repeating decimal. Consequently, the product must be cyclic – of the same six digits in the period for 1/7. This can also be observed by dividing different powers of 10 by 7 and noting down the remainders. If 10 is divided by 7, the remainder is 3. Symbolically, we represent it as $10^1 \equiv 3 \pmod 7$. It is read as *10 is congruent to 3 modulo 7*. For higher powers of 10, we observe: $10^2 \equiv 2 \pmod 7$, $10^3 \equiv 6 \pmod 7$, $10^4 \equiv 4 \pmod 7$, $10^5 \equiv 5 \pmod 7$, $10^6 \equiv 1 \pmod 7$, $10^7 \equiv 3 \pmod 7$, etc. Observe that the remainders started repeating from 7 onwards. In other words, the remainders repeat with period 6, the maximum period for 7.

At this point, it may be interesting to note that every prime number is not a cyclic prime. Take an example of 37: 1/37 is $0.\overline{027}$. Definitely, it is not a full-period prime and hence, does not generate a cyclic number. Cyclic numbers are generated by the following primes: 7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, 167, 179, 181, 193, ... (A001913 in OEIS). A general method for finding cyclic primes is not known. First few cyclic numbers are: 142857 (1/7), 0588235294117647 (1/17), 052631578947368421 (1/19), … (A180340 in OEIS).

The numbers of cyclic numbers $\leq 10^n$ for n = 1, 2, 3, 4, 5, ... are 1, 9, 60, 467, 3617, 25883, ... It has only been conjectured, but yet to be proved, that there exists an infinite number of cyclic numbers (Gardner, 1992). The fraction of primes which generate cyclic numbers is roughly 3/8, and Emil Artin (1898-1962) conjectured that it is actually the following number (Wrench, 1961):

$$C = \prod_{k=1}^{\infty} \left[1 - \frac{1}{p_k \left(1 - p_k\right)} \right] = 0.3739558136...$$

This number is known as *Artin's constant* (here p_k denotes the k^{th} prime number). It may be noted that the fraction of cyclic numbers among primes less than 10^{10} is 0.3739551, very close to Artin's constant.

Let us look at what happens when the prime is not cyclic. Take the example of 13. It may be noted that $1/13 = 0.\overline{076923}$, the period length is 6 and not 12. Therefore, this is not a cyclic number. However, let us still proceed with multiplying 076923 by different natural numbers up to 12. Check the box (→). Can you see the pattern!

076923 × 1 =	076923	
076923 × 2 =		153846
076923 × 3 =	230769	
076923 × 4 =	307692	
076923 × 5 =		384615
076923 × 6 =		461538
076923 × 7 =		538461
076923 × 8 =		615384
076923 × 9 =	692307	
076923 × 10 =	769230	
076923 × 11 =		846153
076923 × 12 =	923076	

The number 076923 cycles for the multipliers: 1, 3, 4, 9, 10, and 12. For the remaining multipliers, the product cycles about another cyclic permutation: 153846. Therefore, instead of one, 13 produces a number with two cyclic classes (Ecker, 1983). The reason that the fractions with denominator 13 come in more than one cyclic class is that, modulo 13, the powers of 10 repeat with period 6: $10^0 \equiv 1 \pmod{13}$, $10^1 \equiv 10 \pmod{13}$, $10^2 \equiv 9 \pmod{13}$, $10^3 \equiv 12 \pmod{13}$, $10^4 \equiv 3 \pmod{13}$, $10^5 \equiv 4 \pmod{13}$, and $10^6 \equiv 1 \pmod{13}$. If the decimal expansion of a prime p is of the order q, then q is a divisor of $p - 1$, and the prime produces a number with $(p - 1)/q$ cyclic classes. Based on the modulo arithmetic, we can state that the order q of the decimal expansion of a prime p is the lowest positive number that satisfies the equation: $10^q \equiv 1 \pmod{p}$. For a full-period prime, q has to be $p - 1$, and in terms of number theory, we say that 10 is the *primitive root* of the prime p.

It may be interesting to note that any cyclic number when multiplied by the prime number, which generates it (7 in the case of 142857), results in a string of 9s: 7 × 142857 = 999999. The interested reader may try to prove this observation.

Cyclic numbers have another strange property–split the cyclic number into two halves (142 and 857, for the cyclic number 142857) and their sum is nothing but 9s (for example, 142 + 857 = 999). This is due to Midy's theorem (Ginsberg, 2004). Midy's theorem states that if the period of a repeating decimal for a/p has an even number of digits, the addition of its two halves results in a sequence of nine. There are prime numbers with even periods but do not produce cyclic numbers, for example 11. They also possess this 9's property. All cyclic numbers are even in length, and therefore, Midy's theorem applies to them.

There is another associate property–split the cyclic number into three parts (if the period length is divisible by 3), namely 14, 28, and 57, and their sum is nothing but 9s: 14 + 28 + 57 = 99. Ginsberg (2004) has proved this generalization of Midy's theorem.

Let me demonstrate another interesting property of cyclic numbers. Let us insert a '9' in the middle of the cyclic number, and another interesting number emerges. For example, by inserting a '9' in the middle of 142857, we obtain 1429857. When it is multiplied by any number from 1 through 6, the product retains the cyclic nature of the original cyclic number with a '9' always in the middle.

$$
\begin{aligned}
142\underline{\mathbf{9}}857 \times 1 &= 142\ \underline{\mathbf{9}}\ 857 \\
\times 2 &= 285\ \underline{\mathbf{9}}\ 714 \\
\times 3 &= 428\ \underline{\mathbf{9}}\ 571 \\
\times 4 &= 571\ \underline{\mathbf{9}}\ 428 \\
\times 5 &= 714\ \underline{\mathbf{9}}\ 285 \\
\times 6 &= 857\ \underline{\mathbf{9}}\ 142
\end{aligned}
$$

Such a number can be created from any cyclic number by inserting as many 9's in the middle. I propose to call them *axial cyclic numbers* (I could not find any information about these numbers from literature. Please inform me if you locate any information about such a number). So, 142**99**857, 142**999**857, … etc. are all axial cyclic numbers. Naturally, you may ask, what happens when any number other than nine is inserted. Or, what happens when nine is inserted at some other place. I'm leaving that for the interested readers to discover themselves.

References

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