A Recreational Problem from Egypt

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The Rhind Papyrus

While dwelling in Egypt, a young Scottish antiquarian and a lawyer by profession, Alexander Henry Rhind (1833-1863) purchased a papyrus. The British Museum acquired this papyrus after his death. It was supposed to be a roll of 18 feet long and 13 inches high (Newman, 1952). Unfortunately, this document was not complete, as some portions were missing. By a stroke of luck, several parts of the missing portions popped up almost fifty years later in the custody of the New York Historical Society. This entire document, well-known since as the *Rhind Papyrus* or *Rhind Mathematical Papyrus*, is a collection of 84 different mathematical problems. Rhind papyrus is the principal source of our knowledge about ancient Egyptians mathematics.

The papyrus was written from right to left and top to bottom in hieratic script (a cursive writing style of ancient Egypt), and not in hieroglyphic script (a kind of pictorial script used in more ancient Egypt). The text is written in two distinct colours—red and black—and is complemented by figures of geometrical shapes. It is inscribed by *A'h-mosè*, commonly known as *Ahmes*. In the title page of the papyrus, Ahmes wrote (Chace, 1927):

> *Accurate reckoning of entering into things, knowledge of existing things all, mysteries… secrets all. Now was copied book this in year 33, month four of the inundation season [under the majesty of the] King of [Upper and] Lower Egypt, 'A-user-Rê,' endowed with life, in likeness of writing of old made in the time of the King of Upper [and Lower] Egypt, '[Ne-ma']et-Rê.' Lo the scribe A'h-mosè writes copy this.*

The first king, *A-user-Rê*, was recognized as a king of the Hyksos (meaning 'rulers of the foreign land') dynasty and ruled somewhere between 1788 BC and 1580 BC (Newman, 1952). The second king, *Ne-ma'et-Rê*, was identified to be Amenem-het III, who ruled during the Middle Kingdom (from 1849 to 1801 BC). We can now estimate the dates of both the previous work and Ahmes' copy with excellent accuracy. However, it is impossible to say whether the older document was itself a copy of earlier work.

The papyrus begins with a division table of two by fifty odd numbers—from 2/3 to 2/101, all expressed as an addition of unit fractions (having one as its numerator). This division table was important as the Egyptians mathematics involve only unit fractions, except for 2/3 and in some very limited cases, 3/4 (Miatello, 2008). In Egyptians mathematics, each fraction should be expressed as the total of such unit fractions. For example, the fraction 7/10 was written as 2/3 1/30. The addition sign (+) is not used explicitly and the unit fractions were places adjacent to each other. The addition of these fractions are implied. Note that the representation is not unique, as 7/10 can also be expressed as 1/5 1/2. Unit fractions continued in use even among Greek mathematicians. Archimedes wrote 1/2 + 1/4 to express 3/4 (Newman, 1952).

Egyptian arithmetic was primarily additive. They used to reduce multiplication and division, similar to the modern computers, to successive additions and subtractions. It was known to the Egyptians that any integer could be uniquely expressed as an addition of terms of the geometrical progression 1, 2, 4, 8,… This is precisely the binary representation of an integer or representation in base 2. To multiply two numbers, they used to express only one of them as a sum of powers of 2. Now, multiply each power of one multiplier by the other number and sum it up to get the result. For example, 12 × 12 can be written as (4 + 8) × 12 = 4 × 12 + 8 × 12 = 48 + 96 = 144. It was written in the following tabular form:

	1	12
	2	24
	\ 4	48
	\ 8	96
Totals	12	144

The symbol ‘\’ was used to demarcate the numbers that sum up to the original multiplier; in this example, 12. The Egyptians, following this process. could perform multiplication of any two numbers simply by repeated doubling and addition.

The method of dividing one number by another was to keep multiplying the divisor until the dividend was reached. While solving problem 24, Ahmes needed to divide 19 by 8. This was executed by multiplying 8 until 19 is reached:

	1	8
	\ 2	16
	$\underline{2}$	4
	\ $\underline{4}$	2
	\ $\underline{8}$	1
Totals	2 $\underline{4}$ $\underline{8}$	19

Problems where you have to determine an unknown quantity, are called *aha* problems. The Egyptians had a method to solve such problems. Let’s see an example. Problem 30 states, “What is the quantity of which 2/3 1/10 will make 10?” (Maor, 1998). In modern notation, it asks to solve the linear equation with a single unknown: (2/3 + 1/10) x = 10. This linear equation was solved by a procedure, called the *rule of false position*. To solve this equation, first assume a suitable value of x, say 30, and substitute this value in the linear equation. The left-hand side results in 23, in its place of the desired 10. Now, 23 should be multiplied by 10/23 to get the correct answer 10. Therefore, the right answer should be 10/23, multiplied by the initially assumed value, 30. In other words, $x = 300/23 = 13 + 1/23$. Almost four millennia ago, Egyptians had a method to solve linear equations.

There are problems to calculate areas of different geometrical shapes such as triangles, trapezoids, rectangles, and circles. Similarly, there are problems to determine volumes of cylinders and prisms. Problem 41 is related to the calculation of the area of a circle. The problem states, "Find the volume of a cylinder of diameter 9 and height 10" (Maor, 1998). The solution, as could be decipher from the papyrus, follows: "Take away 1/9 of 9, namely 1; the remainder is 8. Multiply 8 times 8, that is, 64. Multiply 64 times 10, it makes 640 cubic cubits" (Maor, 1998). Essentially, the previous procedure corresponds to the area of the circular base with diameter d, as $A = [(8/9)\ d]^2$. Now, compare this formula with the modern formula for calculating the area of a circle $A = \pi\ d^2/4$. We can conclude that Egyptians used the value π as 256/81 = 3.16049…, an error of only 3 in 500 (i.e., 0.6%). Remarkable!

A Recreational Problem

Problem 79 of the Rhind Papyrus cryptically states (Maor, 1998):

A house inventory:		houses	7
1	2,801	cats	49
2	5,602	mice	343
4	11,204	spelt	2,301
		hekat	16,807
Total	19,607	Total	19,607

This entry of 2301 for spelt was erroneous. The correct entry should be 2,401. We can identify a geometric progression with 7 as the initial item as well as the common ratio. We may even add a little story to this problem. There is a small village of seven houses. Each house has seven cats. Each cat can kill seven mice. Each mouse consumes seven spelt. Each spelt contains seven hekat of grains. Now, determine the total number of items involved. This problem has no apparent practical use and has no connection with other problems in the papyrus, obviously meant to be a puzzle (Spalinger, 1990), a recreational problem for the readers.

The right column of the table provides the terms of the geometric progression, 7, 7^2, 7^3, 7^4, 7^5 and finally, the total of the series, 19,607. In the left column of the table, Ahmes demonstrates a procedure to calculate the answer in a quicker way. A property of geometrical progression was familiar to the Egyptians. Consider a geometric progression with the identical initial term and the common ratio. The addition of the first n terms of this series can also be expressed as the product of the common ratio and one plus the sum of the first $(n - 1)$ terms. In modern mathematical notation, $a + a^2 + a^3 + \ldots + a^n = a(1 + a + a^2 + a^3 + \ldots + a^{n-1})$. To find the sum of 7 + 49 + 343 + 2,401 + 16,807, Ahmes put it as 7 × (1 + 7 + 49 + 343 + 2,401). The sum of the terms inside the parenthesis is 2,801, the first entry of the left-hand column. The second and third terms 5,602, and 11,204 are double of 2,801, and 5,602, respectively. The sum of these terms is 19,607, exactly equals to the multiplication of 7 with 2,801.

Note the similarity of this problem with a familiar 18th century Mother Goose riddle-rhyme.

As I was going to St. Ives
I met a man with seven wives.
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kits.
Kits, cats, sacks and wives,
How many were going to St. Ives?

Depending on the interpretation, the correct answer to this riddle is one or none. Definitely, different from the solution to the puzzle of Ahmes. We should not stretch our imaginations and claim that this Mother Goose rhyme has 4000 years of history.

References

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