Perfect Numbers

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Natural numbers influenced the cultural development of human civilization. In ancient times, based on the sum of the divisors of the natural number, the Pythagoreans classified them as *deficient*, *abundant*, and *perfect* (Honsberger, 1973). Consider 16. The proper divisors of 16, excluding 16 itself, are 1, 2, 4, and 8. Proper divisors of a number, excluding the number itself but including unity, are termed as the *aliquot divisors*. The addition of these aliquot divisors is 15. As the sum is less than 16, they called 16 a deficient number. Now, consider 18. The aliquot divisors of 18, which are 1, 2, 3, 6, and 9, add up to 21. As the total is more than 18, they called 18 an abundant number. On the other hand, the number 6 is called a perfect number as the sum of its aliquot divisors (i.e., 1, 2, and 3) is 6, the number itself. A natural number, $n > 1$, is defined as a perfect number if the sum of all its aliquot divisors equals to itself. It turns out that there are many deficient and abundant numbers, but perfect numbers are scarce. It is not easy to achieve perfection!

From antiquity, it is known that 6, 28, 496, and 8128 are the first four perfect numbers. Symbolically, a perfect number (n) may be represented as $\sigma(n) = 2n$, where $\sigma(n)$ represents the sum of all the divisors of a number n, including itself. For example, divisors of 28 are 1, 2, 4, 7, 14, and 28 and the addition of these divisors, that is $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28$.

The sum-of-the-divisors of a number n, or $\sigma(n)$, is a *multiplicative function*. Multiplicative functions, such as $\sigma(n)$, possess an interesting multiplicative property. Whenever two numbers are *prime to each other*, the functional value of their product is the same as the product of their individual functional values. Two numbers, m and n, are called prime to each other when they do not have any common divisor, other than unity. Thus, $\sigma(m \times n) = \sigma(m) \times \sigma(n)$, whenever m and n are prime to each other. Since any natural number can be expressed uniquely as the product of different primes, it is sufficient to know the value for primes only for multiplicative functions. For any prime p, divisors of p^α are $1, p, p^2, p^3, \ldots, p^\alpha$. Now, their sum can be calculated easily. Therefore, $\sigma(p^\alpha) = 1 + p + p^2 + p^3 + \ldots + p^\alpha = (p^{\alpha+1} - 1)/(p - 1)$. Because of the multiplicative property of sum-of-the-divisor function, for any number, $n = \prod_i p_i^{\alpha_i}$, $\sigma(n) = \sigma\left(\prod_i p_i^{\alpha_i}\right) = \prod_i \sigma\left(p_i^{\alpha_i}\right) = \prod_i \left(\frac{p_i^{\alpha_i+1} - 1}{p_i - 1}\right)$.

In *Elements* (Book IX, Proposition 36), Euclid proved that, if $M_q = 2^q - 1$ is a prime, then $n = 2^{q-1}(2^q - 1)$ is a perfect number (Wells, 2005). This can be proved easily. The sum-of-the-divisors of n is the product of the sum-of-the-divisors of 2^{q-1} and $(2^q - 1)$ as they are prime to each other. Now, the sum-of-the-divisors of 2^{q-1} is $(2^q - 1)$, and the sum-of-the-divisors of $(2^q - 1)$ is 2^q. Their product is exactly double the original number. This argument can be expressed mathematically as: $\sigma(n) = \sigma[2^{q-1} (2^q - 1)] = \sigma(2^{q-1}) \times \sigma(2^q - 1) = (2^q - 1) \times 2^q = 2n$. Primes of the form $M_q = 2^q - 1$ are called *Mersenne primes*, named after the natural philosopher of the seventeenth century, Father Marin Mersenne (1588 – 1648).

Leonhard Euler (1707 – 1783) proved the converse of Euclid's statement: every even perfect number is of the form specified by Euclid. Suppose the even number n is perfect. Then n may be written as $2^p \times k$, where k is odd and $p \geq 1$. Now, $\sigma(n) = 2n$ implies $\sigma(2^p k) = \sigma(2^p) \times \sigma(k) = (2^{p+1} - 1)\, \sigma(k) = 2^{p+1} \times k$. Therefore, $\sigma(k) = 2^{p+1} \times k/(2^{p+1} - 1) = k + k/(2^{p+1} - 1)$. Since $\sigma(k)$ is an integer, $(2^{p+1} - 1)$ must be a divisor of k, and this implies that $k/(2^{p+1} - 1)$ is also a divisor of k. As $\sigma(k)$ denotes the sum of all divisors of k, k must have only two divisors, and the latter one must be unity. This implies that k must be prime of the form $2^{p+1} - 1$ and proves the result. Therefore, all even perfect numbers are of Euclidean form, and our knowledge about even perfect numbers depends solely on Mersenne primes.

If a number of the form $M_q = 2^q - 1$ is a prime, then necessary q is a prime number. If m divides q, the $(2^m - 1)$ is a factor of $(2^q - 1)$. It was already known, at Mersenne's time, that not all Mersenne numbers are prime. While $M_2 = 3$, $M_3 = 7$, $M_5 = 31$, $M_7 = 127$ are primes, M_{11} is composite (= 23 × 89). Now, M_2, M_3, M_5, and M_7 are related to the first four perfect numbers: 6, 28, 496, and 8128. In 1644, Mersenne, in the preface to *Cogitata Physico-Mathematica*, wrote that M_q is prime for q = 13, 17, 19, 31, 67, 127, 257 (Bell, 1951). Mersenne was incorrect about 67 and 257 and did not include 61, 89, 107 (among those less than 127), which also produce Mersenne primes. In 1903, Frank Nelson Cole (1861 – 1927) took "three years of Sundays" to produce the factors of M_{67} (Bell, 1951). Bell (1951) recalled an interesting story:

> *"At the October, 1903, meeting in New York of the American Mathematical Society, Cole had a paper on the program with the modest title* On the factorisation of large numbers. *When the chairman called on him for his paper, Cole ... proceeded to chalk up the arithmetic for raising 2 to its sixty-seventh power. Then he carefully subtracted 1. Without a word he moved over to a clear space on the board and multiplied out, by longhand,*
>
> *193,707,721 × 761,838,257,287.*
>
> *The two calculations agreed. ... For the first and only time on record, an audience of the American Mathematical Society vigorously applauded the author of a paper delivered before it. Cole took his seat without uttering a word. Nobody asked him a question."*

In 1922, Maurice Kraitchik showed that M_{257} is composite without finding any actual factor (Cohen, 1976). Even with these few mistakes, Mersenne's statement is pretty astonishing, especially considering the size of the numbers involved.

In 1814, Peter Barlow opined (Wells, 1997), "*Euler ascertained that* $2^{31} - 1 =$ 2,147,483,647 *is a prime number; … and probably the greatest that ever will be discovered; … it is not likely that any person will attempt to find one beyond it*." Barlow misjudged the passion of number-crunchers, powered with modern computers. Prime numbers that produce Mersenne prime are 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, … (A000043 in OEIS). To date, 51 Mersenne primes and hence, 51 even perfect numbers are known. The largest Mersenne prime, $M_{82589933}$ with 24862048 digits, was discovered on 21 December 2018. For the latest update, you may follow Great Internet Mersenne Prime Search (GIMPS).

Even perfect numbers have some interesting properties. They can be written as a sum of consecutive natural numbers implying that the even perfect numbers are triangular. Except 6, other even perfect numbers can also be written as the sum of the cube of consecutive odd integers.

$6 = 1 + 2 + 3$

$28 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 1^3 + 3^3$

$496 = 1 + 2 + 3 + ... + 31 = 1^3 + 3^3 + 5^3 + 7^3$ and so on.

This is so because

$$2^{q-1}\left(2^q - 1\right) = \sum_{k=1}^{(q-1)/2} \left(2k-1\right)^3 = \sum_{i=1}^{2^q-1} i \,.$$

At this point, you may enquire about odd perfect numbers. I'm sorry. I cannot provide any example. In fact, not even one has ever been found. An example of an odd perfect number has been comprehensively searched, but it is still elusive. If one exists, it must be larger than 10^{1500} (Ochem and Rao, 2012) with an enormous number of prime factors. The existence of an odd perfect number '*stands like an unconquerable fortress*.' Interestingly, as observed by Harold Shapiro (2008), '*not even a wrong proof has ever been published*.'

References

Bell, E.T., 1951. Mathematics: Queen and Servant of Science. Bell & Sons, London.

Cohen, D.B., 1976. Another Way of Finding Prime Numbers. *The Mathematics Teacher*, 69(5), 398-400.

GIMPS, Great Internet Mersenne Prime Search (https://www.mersenne.org/).

Honsberger, R., 1973. Mathematical Gems: From Elementary Combinatorics, Number theory, and Geometry, Mathematical Association of America.

OEIS, The On-Line Encyclopedia of Integer Sequences (https://oeis.org/).

Ochem, P. and Rao, M., 2012. Odd Perfect Numbers Are Greater Than 10^{1500}. *Mathematics of Computation*, 81(279), 1869-1877.

Shapiro, H.N., 2008. Introduction to the Theory of Numbers. Dover, New York.

Wells, D., 1997. The Penguin Dictionary of Curious and Interesting Numbers. Penguin, London.

Wells, D.G., 2005. Prime Numbers: The Most Mysterious Figures in Math. John Wiley & Sons, Hoboken, New Jersey.